TEST 1 (Complex Numbers & Vectors)

Worth 5% of the Year Mark 50 minutes permitted.

Name : SOLUTIONS

Score : (out of 60)

1. [10 marks]

Given complex numbers **z** and **w** where $\mathbf{z} = 3+5\mathbf{i}$ and $\mathbf{w} = 4-7\mathbf{i}$

(a) Determine, exactly

2m (i) $|\mathbf{z}-\mathbf{w}| = |-1+12i| = \underline{\underline{\sqrt{145}}}$ ✓ ✓

[2]

1m (ii) $\text{Re}(\mathbf{z}) - \text{Im}(\mathbf{w}) = 3-(-7)$

$= \underline{\underline{10}}$ ✓

[1]

3m (iii) $\dfrac{1}{\overline{z}-\overline{w}} = \dfrac{1}{-1-12i} \times \dfrac{-1+12i}{-1+12i}$ ✓ ✓

$= \dfrac{-1+12i}{\sqrt{145}}$

$= \underline{\underline{\dfrac{-1}{\sqrt{145}} + \dfrac{12}{\sqrt{145}}i}}$ ✓

[3]

(b) Find the value of a such that $a\mathbf{z} + 3\mathbf{w} = 6 - 31\mathbf{i}$

4m

$3a + 5ai + 12 - 21i = 6 - 31i$ ✓

$\therefore 3a + 12 = 6$ ✓ (comparing real parts) ✓

$3a = -6$

$\underline{\underline{a = -2}}$ ✓

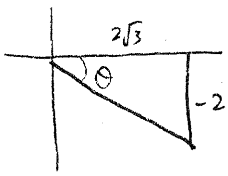
[4]

10

2. **[8 marks]**

Consider the complex numbers $\mathbf{u} = 2\sqrt{3} - 2\mathbf{i}$ and $\mathbf{v} = \mathbf{i} - 1$

3m (a) Write $\mathbf{u}$ and $\mathbf{v}$ in exact polar form.



$u = 4 \text{ cis} \left(-\frac{\pi}{6}\right)$

$v = \sqrt{2} \text{ cis} \left(\frac{3\pi}{4}\right)$

$\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore \theta = -\frac{\pi}{6}$

[3]

3m (b) Simplify $\frac{u^2}{v^6}$, leaving your answer exactly in polar form.

$$\frac{16 \text{ cis}\left(\frac{-\pi}{3}\right)}{8 \text{ cis}\left(\frac{9\pi}{2}\right)} = \frac{16 \text{ cis}\left(-\frac{\pi}{3}\right)}{8 \text{ cis}\left(\frac{\pi}{2}\right)}$$

$$= 2 \text{ cis}\left(\frac{-5\pi}{6}\right)$$

[3]

2m (c) Find exactly $|\mathbf{u} + 2\mathbf{v}|$

$$|2\sqrt{3} - 2i + 2i - 2|$$

$$= |2\sqrt{3} - 2|$$

$$= 2\sqrt{3} - 2$$

[2]

8

3. **[9 marks]**

(a) Sketch the graphs in the Argand Plane to indicate the set of numbers z that satisfy :

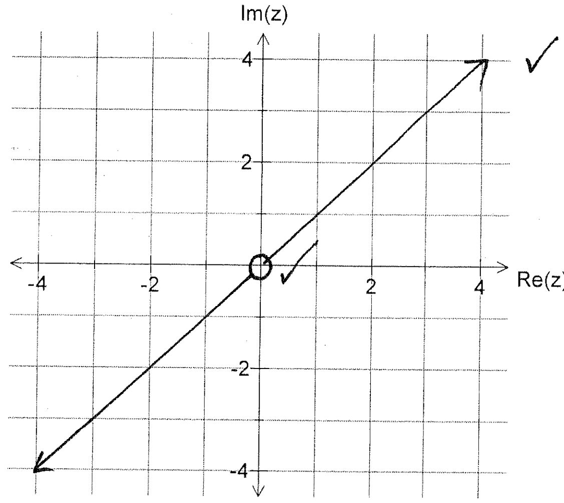
3m (i) $\frac{z}{\bar{z}} = i$

let $z = a + bi$

$a + bi = i(a - bi)$

$a + bi = b + ai$

$\therefore a = b$

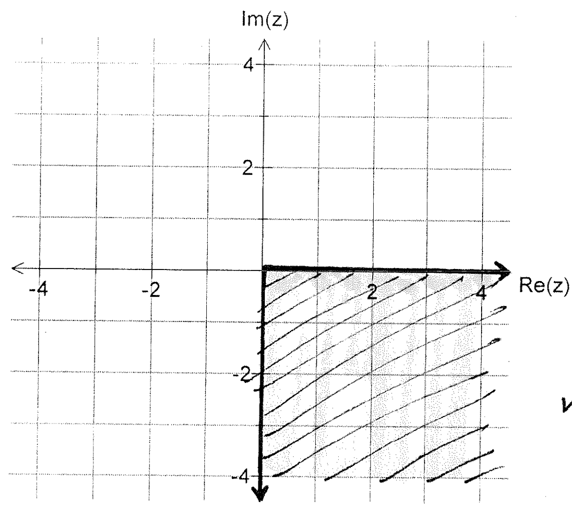


[3]

3m (ii) $-\frac{\pi}{6} \le Arg\left[(1+\sqrt{3}\,i)z\right] \le \frac{\pi}{3}$

$-\frac{\pi}{6} \le Arg(z) + \frac{\pi}{3} \le \frac{\pi}{3}$

$-\frac{\pi}{2} \le Arg(z) \le 0$



[3]

3m

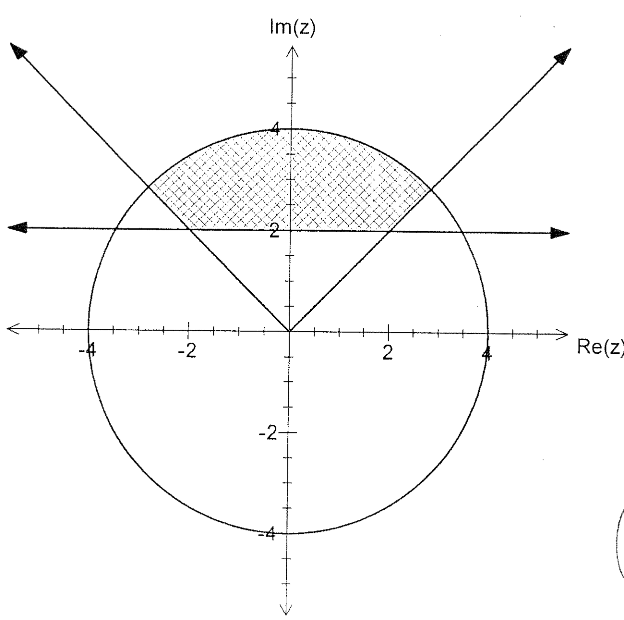
(b) Describe the shaded region in the Argand plane below.

[3]

$|z| \le 4$

$Im(z) \ge 2$

$\frac{\pi}{4} \le Arg(z) \le \frac{3\pi}{4}$

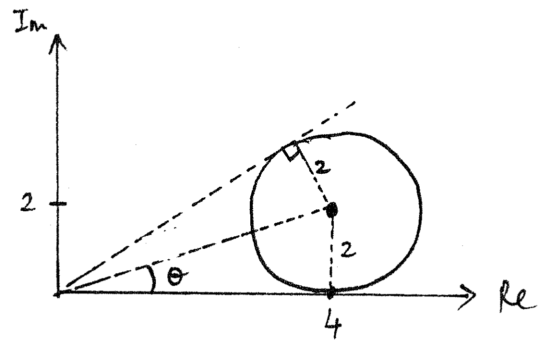


9

4. **[5 marks]**

For the region in the Argand plane defined by the inequality $|z - 4 - 2i| \le 2$,

determine the maximum and minimum value for the argument of z.



✓

$\underline{\underline{\text{Min } Arg(z) = 0}}$ ✓

$\tan\theta = \frac{1}{2}$ ✓

$\therefore \theta = \tan^{-1}\left(\frac{1}{2}\right)$

$= 26.565°$ or 0.464^r (3 dp) ✓

$\therefore$ $\underline{\underline{\text{max } Arg(z) = 53.13°}}$ or 0.93^r (2 d.p.) ✓

5

5. **[5 marks]**

(a) State the geometrical relationship between the complex numbers **w** and **z** if it is known that $\mathbf{w} = i\mathbf{z}$

2m

w is a rotation $\frac{\pi}{2}$ anti-clockwise about the origin. ✓

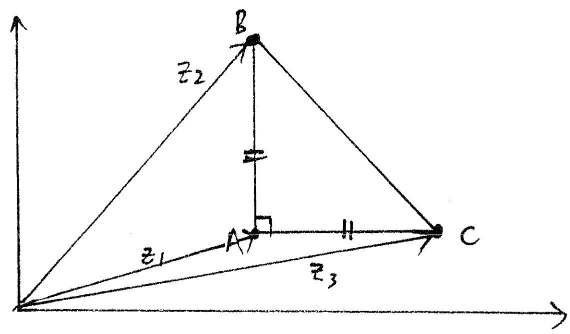
Also, $|w| = |z|$. ✓

[2]

(b) The three points A, B and C in the Argand plane correspond to complex numbers $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ respectively. The triangle ABC is isosceles and has a right angle at A.

3m

Write down algebraically the relationship between $\mathbf{z}_3 - \mathbf{z}_1$ and $\mathbf{z}_2 - \mathbf{z}_1$.
Explain how you arrived at your answer.



✓

$z_3 - z_1 = \vec{AC}$

$z_2 - z_1 = \vec{AB}$

since $\vec{AB} \perp \vec{AC}$, ✓

[3]

$$\underline{\underline{(z_3 - z_1) = i\,(z_2 - z_1)}}$$ ✓

5

$$\boxed{\text{Also accept} \quad z_2 - z_1 = i\,(z_3 - z_1)}$$

6. **[23 marks]**

Consider the following vectors in space :

$$\mathbf{a} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} x \\ 3 \\ -2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ -z \end{pmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$$

Determine :

1m (a) vector **e** such that **e** is parallel to **d** and double its length.

$$\underset{\sim}{e} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix} \checkmark$$

[1]

3m (b) the acute angle between vectors **a** and **d** (to nearest degree).

use dot products :

$$-2 + 15 = (\sqrt{17})(\sqrt{26}) \cos\theta \checkmark$$

$$\cos\theta = \frac{13}{(\sqrt{17})(\sqrt{26})}$$

$$\therefore \theta = 52° \checkmark \text{ (nearest degree)}$$

[3]

2m (c) the relationship between x and z if **c** is perpendicular to **b**.

$$\underset{\sim}{c} \cdot \underset{\sim}{b} = 0 \checkmark$$

$$5x + 2z + 6 = 0 \checkmark$$

[2]

3m (d) the value of x such that **a** is parallel to **b**.

$$\begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} = k \begin{pmatrix} x \\ 3 \\ -2 \end{pmatrix} \checkmark$$

$2 = 3k$, $k = \frac{2}{3}$ and $-3 = -2k$, $k = \frac{3}{2}$ ✓ } since k is not unique, No value of x will make $\underset{\sim}{a} \parallel \underset{\sim}{b}$. ✓

[3]

3m (e) vector **f** such that **f** is in the direction of **a** with a magnitude of 17 units.

$$\underset{\sim}{f} = \frac{17}{\sqrt{17}} \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} \checkmark$$

$$= \sqrt{17} \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2\sqrt{17} \\ 2\sqrt{17} \\ -3\sqrt{17} \end{pmatrix} \checkmark$$

[3]

12

(f) a vector which is perpendicular to both **a** and **d**.

4m

let the vector required be $\underset{\sim}{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\therefore \left.\begin{aligned} -2a + 2b - 3c &= 0 \\ a \qquad\quad - 5c &= 0 \end{aligned}\right\} \checkmark$$

$$\therefore \quad a = 5c \quad \checkmark$$

then, $-10c + 2b - 3c = 0$

$$b = \frac{13c}{2} \quad \checkmark$$

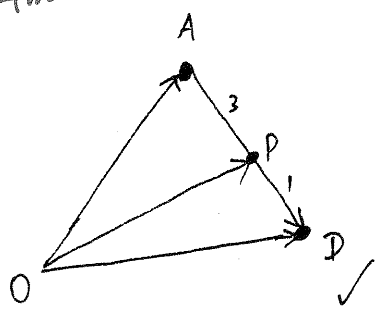
so, choose $c = 2$, then $a = 10$ and $b = 13$ $\Rightarrow$ $\underset{\sim}{v} = \begin{pmatrix} 10 \\ 13 \\ 2 \end{pmatrix}$ ✓

[4]

Suppose that vectors **a** and **d** represent position vectors of points A and D respectively.

(g) Determine the position vector **p** for the point P which divides $\overrightarrow{AD}$ internally in the ratio 3:1.

4m



$$\begin{aligned} \vec{OP} &= \underset{\sim}{a} + \tfrac{3}{4}\vec{AD} \quad \checkmark \\ &= \underset{\sim}{a} + \tfrac{3}{4}\underset{\sim}{d} - \tfrac{3}{4}\underset{\sim}{a} \\ &= \tfrac{1}{4}\underset{\sim}{a} + \tfrac{3}{4}\underset{\sim}{d} \quad \checkmark \\ &= \tfrac{1}{4}\begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \tfrac{3}{4}\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ -\frac{9}{2} \end{pmatrix} \quad \checkmark \end{aligned}$$

[4]

(h) Determine the vector equation for the line in space that connects points A and D.

3m

$$\begin{aligned} \underset{\sim}{r} &= \underset{\sim}{a} + \lambda(\underset{\sim}{d} - \underset{\sim}{a}) \quad \checkmark \\ &= \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad \checkmark \checkmark \end{aligned}$$

11

[3]

End of Test